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REMARKS ON KLEIN'S "FAMOUS PROBLEMS OF ELEMENTARY GEOMETRY."

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No teacher of geometry in a high school should be ignorant of the contents of this little book which appeared in German nearly twenty years ago, but which is also available in an English translation.[1] We here find: (1) the exact statement of the proof of the necessary and sufficient conditions for constructions with ruler and compasses; (2) the proofs of the impossibility of solution, by means of ruler and compasses, of the famous problems: trisection of an angle, duplication of the cube, squaring the circle; (3) full discussion of Gauss's results concerning regular polygons constructible with ruler and compasses; (4) general considerations on algebraic constructions, the integraph, and the geometric construction of π, and other facts of interest in elementary geometry.

While reading this book with a class in geometry for teachers, I had occasion to criticize and elaborate certain parts. In somewhat condensed form, my notes in this connection are given in the following pages.

Gaussian Polygons.—Up to the time of Gauss, no one suspected that it was possible to construct, with ruler and compasses, regular polygons other than those the number of whose sides could be expressed in one of the forms: 2^n, $2^n \cdot 3$, $2^n \cdot 5$, $2^n \cdot 15$. All of these were known to the Greeks. But Gauss proved as early as 1801[2] that whenever a prime number p could be expressed in the form $2^{2^\mu} + 1$, the construction of a regular polygon with p sides was possible by Euclidean methods. It was then apparent that regular polygons not included in the Euclidean series, namely of 17, 257, 65537, . . . sides, could be constructed

[1] F. KLEIN, (a) *Vorträge über ausgewählte Fragen der elementar Geometrie,* ausgearbeitet von F. Tägert. Leipzig, Teubner, 1895, 66 pp. +2 Tafeln. (b) *Famous Problems of Elementary Geometry* . . . authorized translation of F. Klein's Vorträge . . . , by Wooster Woodruff Beman and David Eugene Smith. Boston, Ginn, 1897, ix+80 pp. (c) *Leçons sur certaines questions de géométrie élémentaire* . . . rédaction Française autorisée par l'auteur, par J. Griess. Paris, Vuibert, 1896, 100 pp. There was also an Italian edition published at Turin in 1896, and a Russian edition which appeared in Kasan in 1898.

[2] *Disquisitiones arithmeticæ,* Lipsiæ, 1801, *Werke,* Bd. 1, 2, abdruck 1870, p. 462; French ed. *Recherches Arithmétiques,* Paris, 1807, p. 488; Ger. ed. by Maser, Berlin, 1889, p. 447.

under the same imposed conditions. And indeed Gauss's discussion led to the result,[1] that the *only* regular polygons which it is possible to construct with ruler and compasses, are those the number P of whose sides can be expressed in the form

$$2^{\alpha} \cdot (2^{2^{\alpha_1}} + 1) \cdot (2^{2^{\alpha_2}} + 1) \cdot (2^{2^{\alpha_3}} + 1) \cdots (2^{2^{\alpha_s}} + 1),$$

where $\alpha_1 \cdots \alpha_s$ are distinct positive integers and each $2^{2^{\alpha_i}} + 1$ is a prime. The number of such polygons is small in comparison with the number of regular polygons which can not be constructed with the means employed. As Professor Dickson has pointed out[2] the number of P's up to 100 is 24; up to 300 is 37 (all noted by Gauss); up to 1000 is 52; up to 100000 only 206.

The determination of the number of regular polygons which can be constructed for P less than a given integer is, then, bound up in the determination of the prime numbers p. Now for only 17 values of μ has it been shown whether p is prime or not, namely for the values of μ from 0 to 9 inclusive, and for 11, 12, 18, 23, 36, 38, 73. In the first five of these cases, and in these alone, is p prime. The proof of these five cases was given by Fermat in the seventeenth century. It may well turn out that p is not prime, for $\mu > 4$, although Eisenstein announced[3] the theorem: "There are an infinity of prime numbers of the form $2^{2^{\mu}} + 1$." The result for the case $\mu = 8$ was announced only some five years ago and that for $\mu = 7$ four years earlier. When Klein stated[4] in 1895 that $\mu = 7$ does not give a prime number, he was merely prophetic, while his further statement that for "$\mu = 8$ no one has found out whether we have a prime number or not," must now be modified.

The results already established in this connection may be set forth in tabular form[5] as follows:

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[1] This result was, in effect, stated but not proved by Gauss. *Cf.* the next section.

[2] L. E. Dickson, "On the Number of Inscriptible Regular Polygons," *Bull. N. Y. Math, Soc.*, Feb., 1894, vol. 3, p. 123.

[3] G. Eisenstein, "Aufgaben," *Crelle's Journal*, Vol. 27, 1844, p. 87.

[4] English edition, p. 16; German edition, p. 13; French edition, pp. 26–27.

[5] Largely as by Cunningham and Western in *Proc. London Math. Soc.*, Vol. 1, p. 175, 1903. *Cf. Encyclopédie des sciences mathématiques*, Tome 1, Vol. 3, 1906, p. 51. The authorities for the different results connected with the corresponding case numbers, except those of Fermat, are as follows:

6. L. Euler, *Commentarii Academiae Scientiarum Petrop.*, 1738, Vol. 6 (1732–3), pp. 103–107 p. 104; according to the "Akten" laid before the Academy of St. Petersburg, 26 Sept. 1732. The case $\mu = 5$ was completely factored by Euler about December, 1729. For other editions of this memoir see G. Eneström, *Verzeichnis der Schriften Leonhard Eulers*. Erste Lieferung, 1910, p. 7.

In his autobiography (Springfield, Mass., 1833, p. 38) the American calculator Zera Colburn records that while on exhibition in London, at the age of 8, he found "by the mere operation of his mind" the factors 641 and 6,700,147 of 4,294,967,297 ($= 2^{32} + 1$). *Cf.* F. D. Mitchell, "Mathematical Prodigies," *Amer. Journal of Psychology*, Vol. 18, 1907, p. 65.

7. Lucas, *Comptes Rendus*, Vol. 85, 1878, p. 138; *Amer. Jour. Math.*, Vol. 1, 1878, p. 238; *Recreations mathématiques*, Vol. 2 (2e éd., 1896), pp. 234–5. Landry, *Nouv. Corresp. Math.*, Vol. 6, 1880, p. 417.

8. Independent discoverers: Western, *Proc. Lond. Math. Soc.*, (2), Vol. 3, pp. xxi–xxii. Abstract of paper read, 1905; Morehead, *Bull. Amer. Math. Soc.*, Vol. 11, pp. 543–545, abstract, of paper read April 29, 1905.

9. Western and Morehead, *Bull. Amer. Math. Soc.*, Vol. 16, 1909, pp. 1–6; "Each doing half of the whole work."

Case.	μ	Factors of p.	Discoverer.	Year.
1–5	0–4	All prime	Fermat	1640
6	5	$\left\{\begin{array}{l}2^7\cdot5+1=641\\2\cdot{}^752347+1=6700417\end{array}\right\}$	L. Euler	1729
7	6	$\left\{\begin{array}{l}\text{Unknown but composite.}\\2^8\cdot9\cdot7\cdot17+1=274177\\2^8\cdot5\cdot52562829149+1\end{array}\right.$	Lucas Landry Landry and Le Lasseur	1878 1880 1880
8	7	Unknown but composite.	A. E. Western, J. C. Morehead	1905
9	8	Unknown but composite	A. E. Western, J. C. Morehead	1909
10	9	$2^{16}\cdot37+1$	A. E. Western	1903
11	11	$\left\{\begin{array}{l}2^{13}\cdot3\cdot13+1\\2^{13}\cdot7\cdot17+1\end{array}\right\}$	A. Cunningham	1899
12	12	$2^{14}\cdot7+1$ .... $\left\{\begin{array}{l}2^{16}\cdot397+1\\2^{16}\cdot7\cdot139+1\end{array}\right\}$	E. Lucas and P. Pervušin A. E. Western	1877 1903
13	18	$2^{20}\cdot13+1$..........................	A. E. Western	1903
14	23	$2^{25}\cdot5+1$..........................	P. Pervušin	1878
15	36	$2^{39}\cdot5+1$..........................	Seelhoff	1886
16	38	$2^{41}\cdot3+1$..........................	J. Cullen, A. Cunningham, A. E. and F. J. Western	1903
17	73	Unknown but composite.	J. C. Morehead	1906

The labor expended in deriving these results has been enormous. And to the layman who knows nothing of congruences in the theory of numbers, the facts found must seem almost to border on the miraculous. For, even when $\mu = 10$, a case not yet solved, p contains 309 digits; but when $\mu = 36$, p is a number of more than twenty trillion digits. Concerning it Lucas remarked[1] " la bande de papier qui le contiendrait ferait le tour de la Terre." For $\mu = 73$, Ball states that the digits in p " are so numerous that, if the number were printed in full with the type and number of pages used in this book [*Mathematical Recreations*, fifth edition, 1911, 508 pages], many more volumes would be required than are contained in all the public libraries of the world."

In not less than seven places[2] did Fermat refer to $2^{2^\mu} + 1$ as an expression for

10, 12 (Western), 13, 16. *Proc. Lond. Math. Soc.* (2), Vol. 1, 1903, p. 175; abstract of paper read May 14, 1903.

11. A. Cunningham, *Brit. Assoc. Rept.*, 1899, pp. 653–4.

12, 14. E. Lucas, *Atti Accad. Torino*, Vol. 13 (1877–8), p. 271 [27 Jan., 1878]. *Mélanges math. astr. acad. Petersb.*, Vol. 5 (1874–81), p. 505, 519 or *Bull. Acad. Pétersb.* (3) vol. 24, 1878, col. 559; (3) Vol. 25, 1879, col. 63; communication by V. Buniakovskij of results, for $\mu = 12$ and 23, found by J. Pervušin, in Nov. 1877 and Feb. 1878.

15. P. Seelhoff, *Zeitschrift math. u. Phys.*, Vol. 31, 1886, p. 380.

17. J. C. Morehead, *Bull. Amer. Math. Soc.*, Vol. 12, 1906, pp. 449–451.

[1] E. Lucas, *Théorie des nombres*, Vol. 1, Paris, 1891, p. 51.

[2] Letter dated August [?] 1640 to Frenicle (*Oeuvres de Fermat*, Vol. 2, 1894, p. 206; letter dated 18 October, 1640, to Frenicle (*Oeuvres*, Vol. 2, 1894, p. 208; *Varia Opera*, Toulouse, 1679, p. 162; Brassine's *Précis*, Toulouse, 1853, pp. 142–3); letter dated 25 December, 1640, to Mersenne (*Oeuvres*, Vol. 2, pp. 212–213); " De solutione problematum geometriconum per curvas simplicissimas et unicuique problematum generi proprie convenientes, Dissertatio tripartita " (*Oeuvres de Fermat*, Vol. 1, 1891, pp. 130–131; French translation, Vol. 3, 1896, p. 120; *Varia Opera*, 1679 [reprint, 1861], p. 115); letter dated 29 August, 1654, to Pascal (*Œuvres de Pascal*,

determining a series of prime numbers. The earliest reference is in a letter of August[?], 1640, to Frenicle. He wrote:

"But here is something which pleases me greatly: it is that I am almost persuaded that numbers of the progression $2^{2^0}, 2^{2^1}, 2^{2^2}, 2^{2^3}, \cdots$, augmented by 1, are prime numbers, as

$$3,\ 5,\ 17,\ 257,\ 65\,537,\ 4\,294\,967\,297,$$

and the following of 20 digits

$$18\,446\,744\,073\,709\,551\,617;\ \text{etc.}$$

I have not an exact demonstration, but I have excluded such a large number of divisors by infallible proofs, and have so many side lights which bear out my thought, that I would find difficulty in convincing myself of error."

In October of the same year Fermat again wrote to Frenicle

"I have not yet demonstrated the exclusion of all divisors in that beautiful proposition which I sent you and which you verified for me with respect to the numbers, 3, 5, 17, 257, 65537, etc For, although I can prove the exclusion of most divisors and show the probability of exclusion for the rest, I am not yet able to demonstrate the necessary truth of the proposition, concerning which however, I have no more doubt at this moment than I had previously. If you have a sure proof you will oblige me by communicating the same to me; for, after that, nothing can keep me back in these matters."

Fourteen years later Fermat had to write to Pascal,

"The demonstration of the proposition is very difficult and I confess to you that I have not yet fully found it; I should not propose that you seek it, had I already reached the goal."

In a somewhat similar vein he demanded a demonstration of Digby in 1658. Indeed, at five different times, last in 1658, Fermat (died 1665) carefully noted that he lacked a rigorous proof that $2^{2^\mu} + 1$ is always a prime number. It is remarkable that he overlooked the fact that this was not prime for $\mu = 5$, since he himself made a remark on the possible factors of numbers of the form $2^m \pm 1$, from which it may be shown that the prime factors of $2^{32} - 1$ must be primes of the form $64n + 1$.[1] From this, Euler's factors 641 and 6700417 could be deduced at once. Fermat's exact statement here, as elsewhere, tends but to confirm the belief in the accuracy of what he wrote with regard to his celebrated theorem (the proof of which has baffled the greatest mathematicians ever since): "I have found for this a truly wonderful proof."[2]

Gauss's Statement of his Polygon Results. In two passages the implication to be drawn from what Klein has written is, that Gauss published a proof that a regular polygon of p sides can not be constructed by ruler and compasses if p is a prime not of the form $2^k + 1$. The passages to which I refer are:[3]

Vol. 4, Paris, 1819, p. 384; *Oeuvres de Fermat*, Vol. 2, 1894, pp. 309–310); letter to Sir Kenelm Digby, sent by Digby to Wallis, 19 June, 1658 (*Oeuvres de Fermat*, Vol. 2, 1894, pp. 402, 404–5; French translation of the Latin, Vol. 3, 1896, p. 314, 316); letter dated August, 1659 to Carcavi, copy sent by Carcavi to Huygens 14 August, 1659 (*Corresp.* de Huygens no. 651; *Oeuvres de Fermat*, Vol. 2, pp. 433–434).

[1] W. W. R. Ball, *Math. Recreations and Essays*, fifth ed., London, 1911, p. 40, is authority for this last statement. I have a distinct impression that it has also been made by some first-class authority, but I have vainly searched Fermat's works for its verification.

[2] *Oeuvres de Fermat*, Vol. 1, 1891, p. 291: "Cujus rei demonstrationem mirabilem sane detexi."

[3] English edition, pp. 2, 16; German edition, pp. 2, 13; French edition, pp. 10, 26.

(1) " Gauss added other cases [to Euclid's] by showing the *possibility* of the division into parts where p is a prime number of the form $p = 2^{2^\mu} + 1$, and the *impossibility* for all other numbers " (the italics here and in (2) are mine); (2) " Gauss extended this series of numbers [Euclid's] by showing that the division is *possible* for every prime number of the form $p = 2^{2^\mu} + 1$ but *impossible* for all other prime numbers and their powers." Now the implication referred to above is not correct, as Professor Pierpont interestingly set forth in his paper " On an undemonstrated theorem of the *Disquisitiones Arithmeticae*."[1] That is, Gauss *did not give a proof* of the " impossibility " referred to in the quotations. But after proving the " possibility " as described above he uses the following words:

"As often as $p-1$ contains other prime factors besides 2, we arrive at higher equations,[2] namely, to one or more cubic equations, if 3 enters once or oftener as a factor of $p-1$, to equations of 5th degree if $p-1$ is divisible by 5, etc. And we can prove with all rigour that these equations cannot be avoided or made to depend upon equations of lower degree; and although the limits of this work do not permit us to give the demonstration here, we still thought it necessary to signal this fact in order that one should not seek to construct other polygons than those given by our theory, as, for example, polygons of 7, 11, 13, 19 sides, and so employ one's time in vain."

To carry Gauss's reasoning further, by supplementing what I have given in the last section, it will be of interest to follow Professor Pierpont. He says:

" Having laid down the theory for polygons of a prime number of sides, Gauss now turns his attention to polygons of any number of sides, $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_\nu^{a_\nu}$ where $p_1, p_2, \cdots$, are the prime factors of n. These he disposes of in a very summary fashion by declaring, without any attempt at proof, that they can be constructed then and only then when

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_\nu}\right)$$

contains no other factor than 2."

" That this is a sufficient condition follows at once from an easy extension of Gauss's method as developed when n is a prime. It is, however, vastly more important to know that *only* these polygons can be geometrically constructed as thereby the theory of regular polygons, as far as their construction by ruler and compasses is concerned, is made complete. That is, in a given case we can decide whether the polygon is constructible, and in case that it is, Gauss's theory gives us the necessary directions to construct it."

In the first part of his paper Professor Pierpont shows " that the condition which Gauss gave as necessary is in fact such."

Geometrical Constructions of the Regular 17-side. The remark of Klein[3] that we possess as yet no method of construction of the regular polygon of seventeen sides, based upon considerations purely geometrical, is a little curious, since several constructions of this kind have been given. One by Erchinger was indeed reported by Gauss in 1825.[4] The construction is as follows:

[1] *Bull. Amer. Math. Soc.*, Dec., 1895, Vol. 2, pp. 77–83. *Cf.* also L. E. Dickson in *Monographs on Topics of Modern Mathematics*, London and N. Y., 1911, p. 386.

[2] In his earlier discussion of an inscribed polygon of p sides, Gauss considers the equation $x^p - 1 = 0$ and the resulting equation got by dividing out the factor $x - 1$ where $p =$ prime.

[3] English edition, pp. 24, 32; German edition, pp. 19, 26; French edition, pp. 35, 43.

[4] *Göttingische gelehrte Anzeigen*, Dec. 19, 1825, no. 203, p. 2025; *Werke*, Vol. 2, pp. 186–7. To Art. 365 of the *Disquisitiones Arithmeticae* Gauss added this note in his handwriting: " Circulum in 17 partes divisibilem esse geometrice, deteximus 1796 Mart. 30." [*Cf. Werke*, Vol. 1, p. 476.

Let D, B, G, A, I, F, C, E be points on a line determined by constructions about to be given. Let AB be a line of any length. Produce it both ways to

Fig. 1.

C and D so that,

$$AC \times BC = AB \times BD = 4\overline{AB}^2.$$

Further determine the points E, G, on both sides of CA produced so that,

$$AE \times EC = AG \times CG = \overline{AB}^2;$$

and find the point F on the side A of the line BA produced, such that

$$AF \times DF = \overline{AB}^2.$$

Finally divide AE in I so that

$$AI \times EI = AB \times AF,$$

where AI is the smaller, and EI the larger part of AE. Then construct a triangle, in which each of two sides equals AB, the third being equal to AI. About this triangle describe a circle; then AI will be one side of the regular inscribed polygon of seventeen sides.

Gauss particularly remarks that the author gave a purely synthetic proof of this construction.

Another synthetic construction and proof dated "Dublin, 17th October, 1819" was published by Samuel James in the *Transactions of the Irish Academy*.[1] The following earlier solution by John Lowry was Prize Question 410 in *The Mathematival Repository* for 1819:[2] "Draw the radius CO at right angles to the

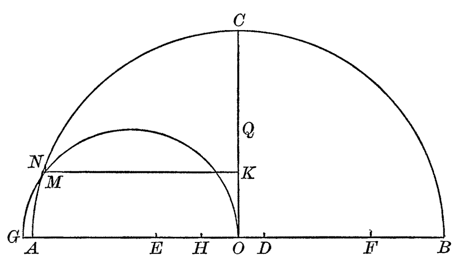

Fig. 2.

diameter AB;[3] on OC and OB, take OQ equal to the half, and OD equal to an eighth part of the radius; make DE and DF each equal to DQ, and EG and FH respectively equal to EQ and FQ; take OK a mean proportional between OH and OQ, and through K draw KM parallel to AB, meeting the semicircle described on OG in M; draw MN parallel to OC cutting the given circle in N, the arc AN is the seventeenth part of the whole circumference."

[1] Vol. 13 (1818), pp. 175–187; paper read Jan. 24, 1820.
[2] New Series, Vol. 4, p. 160. Lowry's proof occupies pp. 160–168.
[3] O is the middle point of AB.

Irrationality of π.—Klein wrote:[1] "After 1770 critical rigour gradually began to resume its rightful place. In this year appeared the work of Lambert: *Vorläufige Kentnisse für die so die, Quadratur . . . des Cirkuls suchen.* Among other matters the irrationality of π is discussed. In 1794 Legendre in his *Éléments de Géométrie* showed conclusively that π and π^2 are irrational numbers." The implication of this note is that Lambert did not discuss the irrationality of π conclusively and that Legendre did. How both of these points of view are essentially incorrect will appear in what follows. Klein was simply reproducing the erroneous statements of Rudio;[2] but after Pringsheim's careful study in 1898,[3] Lambert's proof emerged as "auserordentlich scharfsinnig und im wesentlichen vollkommen einwandfrei," while Legendre's remained "in Bezug auf Strenge hinter Lambert weit zurück."

As in the later proof of the transcendence of π, so here when its irrationality was in question, discussion of e is fundamental. The irrationality of e and e^2 was substantially shown by Euler in 1737[4] and he gave the expression for e as a continued fraction on which Lambert's proofs of the irrationality of e^x, $\tan x$ and π rest. Starting with Euler's development[5]

$$\frac{e-1}{2} = \frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+}\ \text{etc.},$$

Lambert found

$$\frac{e^x-1}{e^x+1} = \frac{1}{2/x+}\,\frac{1}{6/x+}\,\frac{1}{10/x+}\,\frac{1}{14/x+}\ \text{etc.},$$

and since

$$\frac{e^x-1}{e^x+1} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \tanh\frac{x}{2} = \frac{1}{i}\tan\frac{ix}{2},\quad \text{if}\quad z = \frac{ix}{2},$$

$$\tan z = \frac{1}{1/z-}\,\frac{1}{3/z-}\,\frac{1}{5/z-}\,\frac{1}{7/z-}\,\frac{1}{9/z-}\cdots.$$

He then proved the theorems:

1. *If x is a rational number different from zero, e^x can never be rational.*

For $x = 1$, we have as special case the irrationality of e.

2. *If z is a rational number different from zero, $\tan z$ can never be rational.*

For $z = \pi/4$, $\tan\ \pi/4 = 1$, and hence as a special case the irrationality of π.

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[1] English edition, p. 59; German edition, p. 46; French edition, p. 72.

[2] F. Rudio: *Archimedes, Huygens, Lambert, Legendre, vier Abhandlungen über die Kreismessung*, Leipzig, 1892, p. 56f. This error is also reproduced by B. Calò in Enriques's *Fragen der Elementargeometrie*, II. Teil, 1907, p. 315; by D. E. Smith in Young's *Monographs on Topics of Modern Mathematics*, 1911, p. 401. The matter was correctly set forth by T. Vahlen in *Konstruktionen und Approximationen*, Leipzig, 1911, pp. 319ff.

[3] A. Pringsheim: "Ueber die ersten Beweise der Irrationalität von e und π." *Sitzungsberichte der mathematisch-physikalischen Classe der k. b. Akademie der Wissenschaften zu München*, Bd. 28, 1899, pp. 325–337.

[4] "De fractionibus continuis," *Comment. acad. de Petrop*, Vol. 9, 1744, p. 108. Presented to St. Petersburg Academy, March, 1737.

[5] L. Euler: *Introductio in analysin infinitorum.* Tomus Primus, Lausannae, 1748, p. 319. This work was finished in 1745; *Cf.* G. Eneström, *Verzeichnis* etc., Erste Lieferung, p. 25.

The part of Lambert's " Vorläufige Kentnisse " to which Klein refers contains some formulae without proof, and no analytical developments, and was rather intended to serve as a popular survey of the treatment of the topic. With it must be considered the scientifically remarkable " Mémoire " of 1767.[1] Here " mit minutiöser Genaugkeit " Lambert proves the convergence of the expression for tan z as a continued fraction. Pringsheim dwells on the " astounding " nature of these considerations at this period in the history of mathematical thought. For of such considerations Legendre was innocent, as well as the great Gauss in his 1812 memoir on hypergeometric series, and others, till a much later period.

" Thus the Lambert memoir contains the *first*, and for many years, the *only* example of what we now consider really rigorous developments of functions as converging continued fractions, in particular, that for tan z given above."

Constructions in General with Ruler and Compasses. While the Greeks built elaborate structures with propositions involving Euclidean methods, Descartes was the first to discuss the question, What geometrical constructions are, and what are not, theoretically possible by these methods? This discussion harks back to the famous, anonymously printed, geometry[2] of 1637, in which the first book was entitled: "The problems which can be constructed by employing circles and straight lines only."[3] Before referring to Klein in this connection, I wish to give some indications of Descartes's argument.[4]

Descartes's Discussion. The book begins with the remark that all the problems of geometry can be reduced to a form in which the only condition of their construction is the determination of the lengths of certain right lines; and that all that is necessary to the determination of the length of a line from sufficient data is the power of interpreting geometrically the five arithmetical operations of addition, subtraction, multiplication, division, and the extraction of roots in such a manner that, the quantities operated on being lengths, the result shall also be a length. This is to be effected by the introduction of the unit of length (if necessary) as a factor or divisor, so as to reduce the construction to finding a fourth proportional to three given lines, or a mean proportional, or several mean proportionals, between two given lines. The actual constructions for multiplication, division, and the extraction of the square root are then given; roots of higher order being reserved for subsequent consideration. Descartes thus disposes of a difficulty which had undoubtedly been felt in the geometric interpretation of expressions above the third degree. But he remarks that any expression which

[1] " Memoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques." Lu en 1767. Printed in 1768 in *Hist. de l'acad. royale des sciences et belles-lettres*, Berlin, Année 1761 (!), pp. 265–322.

[2] Discours *de la methode pour bien conduire sa raison, et chercher la verite dans les sciences. Plus La Dioptrique, Les Meteores et La Geometrie, qui sont des essais de cette Methode.* A Leyde, 1637; in *Oeuvres* de Descartes publiès par Charles Adam et Paul Tannery, Tome VI, Paris, 1902, pp. 1–418.

[3] Adam-Tannery edition, *l. c.*, pp. 369–387.

[4] I follow Professor J. M. Peirce closely in part of his article, "References in Analytic Geometry," *Harvard University Library Bulletin*, Nos. 8, 10, 11, 1878–1879, pp. 157–158, 246–250, 289–290.

denotes a line must be homogeneous and of the first degree, if its value is independent of the choice of the unit. He next speaks of the formation and reduction of equations, in the solution of a problem, and observes that those problems are *plane,* that is, can be solved by straight lines and circles on one flat surface, when the final equation, containing one unknown quantity, is of a degree not higher than the second. The reason of this, of course, is that two circles, or a right line and a circle, intersect in only two points. The actual constructions for the different forms of a quadratic equation are then given. It is important to note that Descartes pays no attention to *negative solutions.* Thus he considers only the three forms,[1] $x^2 = +ax + b^2$, $x^2 = -ax + b^2$, $x^2 = +ax - b^2$, disregarding altogether the form $x^2 = -ax - b^2$, and he constructs only *one* root for each of the first two equations, while he constructs two for the third.[2] Indeed, Descartes seems to have reached a less advanced point on this subject than had already been attained by an earlier writer. Albert Girard, a Dutchman, in his *Invention Nouvelle en l'Algèbre* (Amsterdam, 1629) lays down the true principle of negative quantities, exactly and broadly.

After giving the four forms of construction of a root of a quadratic equation, Descartes makes the important remark that *all the problems of ordinary geometry can be constructed by what is contained in these four figures.* And he continues: "I do not believe that this fact was noted by the ancients; for otherwise they had not taken the trouble to write so many large works where the order alone of their propositions tells us that they did not have the true method to discover them all, but that they have simply collected those which they have met."

As a decisive test of the power of his geometry, Descartes next takes up a problem which Pappus of Alexandria cites at the beginning of the 7th book of his *Mathematical Collections,* stating that neither Euclid nor Apollonius nor any other had been able to give a general solution of it. This problem may be stated, in effect, as follows: "Given in any plane n right lines, and also either n or $(n - 1)$ other right lines, to find the locus of a point such that the product of its distances from the first set of lines measured on lines making any given angles with them, shall be in a given ratio to the product of the distances of the point from the second set of lines measured on lines making given angles with them." With the discussion of this problem, the first book of Descartes's Geometry closes.

Klein's Discussion. In the "Introduction" to Klein's book he announces the fundamental problem stated in the question at the beginning of this section. He

[1] Descartes does not use the sign =, but a sign which is perhaps an abbreviation of "aequatur."

[2] In fact, there is no evidence that Descartes perceived that the negative root of an equation (the *false* roots as he calls them) could have any real meaning in a geometric construction; and it is a familiar observation that they generally give a solution of the problem from which they spring, of a kind not contemplated at the outset, and inadmissible if the problem is rigidly interpreted. Later, indeed, in giving constructions for equations of the third and fourth degrees, he exhibits the false roots as well as the true; but in these cases, either of the two sorts of lines may be true roots, according to the sign of one of the coefficients in the equation. Moreover, though Descartes was undoubtedly aware that opposite signs always corresponded to opposite directions, and repeatedly speaks of the possible variations that his constructions thus admit, he does not seem to have apprehended this fact as a *principle,* which might be laid down, once for all, at the outset of the discussion; and he never uses a letter which denotes a quantity as admitting a negative value.

then remarks: "To define sharply the meaning of the word 'construction,' we must designate the instruments which we propose to use in each case. We shall consider," he continues, " (1) straight edge and compasses, (2) compasses alone, (3) straight edge alone, (4) other instruments used in connection with straight edge and compasses."

After two very compact paragraphs, Klein states the following fundamental theorem: " *The necessary and sufficient condition that an analytic expression can be constructed with a straight edge and compasses is that it can be derived from the known quantities by a finite number of rational operations and square roots.*" When this is considered with the following theorem of Chapter I, we have a criterion for identifying the problems under consideration: "*If x, the quantity to be constructed, depends only upon rational expressions and square roots, it is a root of an irreducible equation $\phi(x) = 0$, whose degree is always a power of 2.*" Whence it is shown that if this degree of an irreducible equation is not a power of 2, it cannot be solved by square roots.

Other References. Now the implications in the discussion leading to these theorems are very numerous and no student has mastered all the principles involved until he has approached the subject from other points of view. Foremost to be recommended is Castelnuovo's article adapted from his Projective Geometry[1] for the second part of *Fragen der Elementargeometrie*[2] by Enriques. The articles by Daniele[3] and Giacomini[4] in this same volume should also be carefully studied. Among other results here indicated we find:

That every problem which can be solved with ruler and compasses can also be solved, (*a*) with compasses alone; (*b*) with ruler alone, if we are given a fixed circle with its center in the plane of construction; (*c*) with a ruler alone whose edges are parallel; (*d*) with a ruler alone, whose edges are not parallel, but converge to a point. Many problems can also be solved (*e*) with a ruler and segment-carrier. Theorem (*a*) was first enunciated and proved by Lorenzo Mascheroni (1750–1800) in his famous book *La Geometria del compasso*, Pavia, 1797.[5] An elegant proof by Adler based upon the method of inversion was given in 1890.[6] Mascheroni constructions are treated in English, by Cayley in a paper of 1885,[7] and by Hobson in his recent Presidential Address[8] before the Mathematical Association.

[1] *Lezioni di geometria analitica e proiettiva*, Roma-Milano, 1905. The article is entitled, "Uber die Lösbarkeit der geometrischen Aufgaben mit den elementaren Instrumenten; Betrachtungen vom Standpunkte der analytischen Geometrie." This and the other articles have also just appeared, in slightly modified form, in *Questioni riguardanti le mathematiche elementari*, raccolte e coordinate da F. Enriques, Volume II, Bologna, 1914. [2] Leipzig, 1907.

[3] "Uber die Lösung der geometrischen Aufgaben mit dem Zirkel."

[4] "Uber die Lösung der geometrischen Aufgaben mit dem Lineal und den linealen Instrumenten: Betrachtungen vom Standpunkte der projektiven Geometrie."

[5] French edition by Carette, Paris, 1798; second edition, 1828. A German translation of the first French edition, "vermehrt mit der Theorie vom Gebrauche des Proportional-Zirkels und mit einer Sammlung zur Uebung von mehr denn 400 rein geom. Sätzen, von J. P. Grüson" was published at Berlin in 1825.

[6] A. Adler, "Zur Theorie der Mascheronischen Konstruktionen," *Situngsberichte der Wiener Akademie*, Bd. 99, Abt. IIa, 1890, p. 910ff.

[7] *Messenger of Mathematics*, vol. 14, 1885, pp. 179–181; *Collected Papers*, Vol. 12, pp. 314–317.

[8] "On geometrical constructions by means of the compass," *Mathematical Gazette*, Vol. 7, March, 1913, pp. 49–54.

To Poncelet (1788–1867) Theorem (*b*) is due,[1] although the result is frequently referred to Steiner who published, in 1833, the little classic, *Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und eines festen Kreises.*[2] But many of the theorems, including the fundamental one, here found, had been already given by Poncelet and Lambert. By omitting to state that the middle point of the given circle must be known[3] Steiner inaccurately formulates the fundamental theorem. In a recent course of lectures Hilbert proved that knowledge of the position of the middle point was essential and suggested as a problem; "How many given circles in a plane are necessary in order to determine with ruler alone, the center of one of them?" In 1912 D. Cauer showed: (1) *If two circles intersect in imaginary points in the finite part of the plane, it is impossible to determine the middle point of either circle with ruler alone;* (2) *A center may be determined if the circles cut in real points, touch, or are concentric.*[3] About the same time J. Grossmann discovered[4] a result which leads us to Theorem

(*f*): Every problem which can be solved with ruler and compasses can also be solved with ruler alone if we are given 3 linearly independent circles, no two concentric, in the plane of construction.

Theorems (*c*) and (*d*) were proved by Adler in 1890.[5] The former is however really implied in Steiner's work referred to above. In the *Foundations of Geometry*[6] by Hilbert, a certain Theorem (*e*) plays a special role and the ideas thus suggested were elaborated by Feldblum in his dissertation.[7] While the theorem and its early applications are due to Hilbert, it is remarkable to find that Lambert introduced the compasses in exactly the same way as the segment-carrier was employed in later times: "Geometrical constructions are all based on the ruler and compasses. . . . The ruler is used simply for drawing straight lines, *and the compasses serve the purpose of marking off lengths on them and acting as carrier, as well as of drawing angles and giving to lines their proper position.*"[8]

To Plato (429–348 B.C.) has been attributed the chief influence in determining that among the instruments which might have been chosen for developing a system of geometry, the ruler and compasses were selected. Already in the tenth century we find the Arabian, Aboûl Wafâ of Bagdad, somewhat limiting, apparently, these means at his disposal by solving problems with ruler as before but with a compass whose arms were open at a constant angle.[9] With such instru-

[1] *Traité des propriétés projectives des figures*, Paris, 1822, pp. 187–190.

[2] An abridgment in French was published by A. Lévy in *Nouv. Annales de Math.*, 1908, pp. 390–409.

[3] Page 68.

[4] *Mathematische Annalen*, Vol. 73, 1912, pp. 90–94. See also Vol. 74, 1913, pp. 462–464.

[5] A. Adler, "Uber die zur Ausführung geometrischer Konstruktionen notwendigen Hilfsmittel," *Sitzungsberichte der Wiener Akademie*, Bd. 99, 1890, Abt. II*a*, p. 846 ff.

[6] Fourth German edition, Leipzig, 1913; English edition by E. J. Townsend, Chicago, 1902.

[7] M. Feldblum, *Ueber Elementar-Geometrische Constructionen.* Diss. Gottingen, Warschau, 1899.

[8] J. H. Lambert, *Beiträge zum Gebrauche der Mathematik und deren Anwendung*, Berlin, 1765, Teil I, pp. 23–24. The latter part of the last sentence quoted has not been translated very literally; the original runs: "und der Zirkel dient, um sie zu fassen und abzutragen, desgleichen auch um Winkel zu ziehen, und den Linien ihre behörige Lage zu geben."

[9] *Woepecke*, "Analyse et extrait d'un recueil de constructions géométriques par Aboûl Wafâ," *Journal Asiatique*, 1855.

ments in the sixteenth century, Cardano, Ferraro, and Tartaglia solved certain problems in Euclid. Without reference to the published solutions of Ferraro and Cardano, Giovanni Battista Benedetti published at Venice in 1553 a treatise entitled: *Resolutio omnium Euclidis problematum, aliorumque ad hoc necessario inventorum, una tantummodo circini data apertura.* For a detailed history of this phase of geometry the reader may consult the monograph by Kutta.[1] In English, I recall but two titles in this connection; the first is a rare pamphlet translated from the Dutch by Joseph Moxon in 1677,[2] the second is an article by the late Dr. J. S. Mackay.[3]

To our series we may now add Theorem

(g): Every problem which can be solved with ruler and compasses, can also be solved with a ruler and one fixed aperture of the compasses.

In concluding this section, it should be remarked with regard to the quotation made above from Klein's Introduction, that his "consideration" of constructions with compasses alone, straight edge alone, and other instruments used in connection with straight edge or compasses, is practically confined to fugitive historical notes on pages 33–34, 47 (English edition).

Concluding Remarks. In the latter part of Chapter IV Klein has made a slight slip (English ed., p. 77; Ger. ed., p. 63; Fr. ed., p. 92). In connection with the equation $y = e^x$, he wrote: " To an algebraic value of x corresponds a transcendental value of y, and conversely." " Conversely " leads us to the statement, to a transcendental value of y corresponds an algebraic value of x. But proof of this has nowhere been given; indeed the result is not true, in general. To correct, delete "conversely" and add: "To an algebraic value of y corresponds a transcendental value of x."

It is to be hoped that the editors of a new edition will be moved

(1) to add a proof that the only regular polygons which it is possible to construct with ruler and compasses are those the number P of whose sides can be expressed in the form

$$2^\alpha \cdot (2^{2^{\alpha_1}} + 1) \cdot (2^{2^{\alpha_2}} + 1) \cdot (2^{2^{\alpha_3}} + 1) \cdots (2^{2^{\alpha_s}} + 1)$$

where $\alpha_1 \cdots \alpha_s$ are distinct positive integers and each $2^{2^{\alpha_i}} + 1$ is a prime; α and *one* of the series $\alpha_1, \alpha_2, \cdots \alpha_s$ may be zero.

(2) To at least indicate the connection with continued fractions, and the actual method of determining a and b of the theorem stated on page 17, " if μ and ν are prime to each other, we can always find integers a and b positive or negative, such that $1 = a\mu + b\nu$."

(3) To add the proof that

[1] W. M. KUTTA, "Zur Geschichte der Geometrie mit constanter Zirkelöffnung," *Nova Acta Abh. der Kaiserl. Leop.-Carol. Deutschen Akademie der Naturforscher*, Vol. 71, Halle, 1897, pp. 71–101.

[2] *Compendium Euclidis Curiosi: or, geometrical Operations. Shewing how with a single opening of the compasses and a straight ruler all the propositions of Euclid's first five books are performed.* London, 1677. Moxon does not tell us who the author of the original Dutch treatise was.

[3] "Solutions of Euclid's Problems, with a rule and one fixed aperture of the compasses, by the Italian geometers of the sixteenth century," *Proc. Edinb. Math. Soc.*, Vol. 5, 1887, pp. 2–22.

$$(x+1)^{p(p-1)} + (x+1)^{p(p-2)} + \cdots + (x+1)^p + 1 = 0$$

will take the form $\chi^{p(p-1)} + p \cdot \chi(x),$

if p is a prime number, and if "$\chi(x)$ is a polynomial with integral coefficients whose constant term is 1" (pages 22–23).

(4) To point out that although the *Conchoid of Nicomedes* is used in the text to trisect an angle, this application of the curve was the discovery of Pappus (about 300 A.D.), and *not* of Nicomedes (about 180 B.C.).[1] Nicomedes used the curve for the duplication of the cube.[2]

(5) To add the indication of proof that a prime number greater than any number we please *exists*. This is needed in the proof of the transcendence of e.

(6) To add, page 61, the proof that $\mathrm{Lim}_{n \doteq \infty}\, x^n/\underline{|n} = 0$.

(7) To make the following corrections:

Page 12, *for* $F(x) = C\cdot[\phi(x)]^\nu$ *read* $F(x) = C_1\cdot[\phi(x)]^\nu$.

Page 14, line 8 should read

$$+ C_1 \sum_{\nu=1}^{\nu=N} \sum_r e^{|k_\nu|} c_r k_\nu^r q_{r,k_\nu} + C_2 \sum_{\nu=1}^{\nu=N'} e^{|l_\nu|} c_r' l_\nu^r q_{r l_\nu} + \cdots = 0.$$

Page 34, *for* "with the straight edge and one fixed circle we can solve every quadratic equation," *read* "with the straight edge and one fixed circle, the center being given, we can solve every quadratic equation for which line-segments corresponding to the coefficients are given."

Page 72, line 7, *for* b^{Np} *read* $b^{N'p}$; *for* l_N *read* $l_{N'}$.

Also in the theorem on page 5, some ambiguity would be avoided by setting $\phi(x) = 0$ for $f(x) = 0$.

Note. Since this article was written I have seen a new portion, Bd. III, Heft 5, of the *Encyklopädie der mathematischen Wissenschaften,* published at Leipzig on June 8, 1914. It contains a section by J. Sommer on "Elementare Geometrie vom Standpunkte der neueren Analysis aus," pages 773–858, and about half of the section, pages 859–962, by M. Zacharias on "Elementargeometrie und elementare nicht-Euklidische Geometrie in synthetischer Behandlung." Many parts of this Heft will be of interest in connection with questions discussed above. As to the Gaussian polygons, reference might have been given to paragraph 22 of L. O. Hölder's section of the *Encyklopädie,* published in 1899 and entitled, "Galois'sche Theorie mit Anwendungen."

ON THE TRISECTION OF AN ANGLE AND THE CONSTRUCTION OF REGULAR POLYGONS OF 7 AND 9 SIDES.

By L. E. DICKSON, University of Chicago.

1. Purpose and Plan of this Note. Frequently a wide-awake student who has learned how to bisect any angle asks if every angle can be trisected and, if not, why not; after learning how to construct regular polygons of 3, 4, 5, 6, 8

[1] Pappus, ed. Hultsch, p. 246.

[2] *Cf.* CANTOR, *Vorlesungen über Geschichte der Math.*, Bd. I, 3 Aufl., 1907, p. 351.